Symplectic and Isometric $SL(2,\mathbb{R})$-invariant subbundles of the Hodge bundle

Artur Avila, Alex Eskin* and Martin Möller †

September 14, 2012

1 Introduction

Suppose $g \geq 1$, and let $\kappa = (\kappa_1, \ldots, \kappa_n)$ be a partition of $2g-2$, and let $\mathcal{H}(\kappa)$ be a stratum of Abelian differentials, i.e. the space of pairs (M,ω) where M is a Riemann surface and ω is a holomorphic 1-form on M whose zeroes have multiplicities $\kappa_1 \ldots \kappa_n$. The form ω defines a canonical flat metric on M with conical singularities at the zeros of ω. Thus we refer to points of $\mathcal{H}(\kappa)$ as *flat surfaces* or *translation surfaces*. For an introduction to this subject, see the survey [Zo].

The space $\mathcal{H}(\kappa)$ admits an action of the group $SL(2,\mathbb{R})$ which generalizes the action of $SL(2,\mathbb{R})$ on the space $GL(2,\mathbb{R})/SL(2,\mathbb{Z})$ of flat tori.

Affine measures and manifolds. The area of a translation surface is given by

$$a(M,\omega) = \frac{i}{2}\int_M \omega \wedge \bar{\omega}.$$

A "unit hyperboloid" $\mathcal{H}_1(\kappa)$ is defined as a subset of translation surfaces in $\mathcal{H}(\kappa)$ of area one. For a subset $\mathcal{N}_1 \subset \mathcal{H}_1(\kappa)$ we write

$$\mathbb{R}\mathcal{N}_1 = \{(M, t\omega) \mid (M,\omega) \in \mathcal{N}_1, \quad t \in \mathbb{R}\} \subset \mathcal{H}(\kappa).$$

Definition 1.1. An ergodic $SL(2,\mathbb{R})$-invariant probability measure ν_1 on $\mathcal{H}_1(\kappa)$ is called *affine* if the following hold:

(i) The support $\mathcal{N}_1$ of ν_1 is an suborbitfold of $\mathcal{H}_1(\kappa)$. Locally in period coordinates (see §2 below), $\mathcal{N} = \mathbb{R}\mathcal{N}_1 \subset \mathbb{C}^n$ is defined by complex linear equations with real coefficients.

(ii) Let ν be the measure supported on $\mathcal{N}$ so that $d\nu = d\nu_1 da$. Then ν is an affine linear measure in the period coordinates on $\mathcal{N}$, i.e. it is (up to normalization) the restriction of Lebesgue measure to the subspace $\mathcal{N}$.

*Research of the second author is partially supported by NSF grants DMS 0244542, DMS 0604251 and DMS 0905912

†Research of the third author is partially supported by ERC grant 257137 "Flat surfaces"

Definition 1.2. We say that any suborbifold $\mathcal{N}_1$ for which there exists a measure ν_1 such that the pair $(\mathcal{N}_1, \nu_1)$ satisfies (i) and (ii) an *affine invariant submanifold.*

Note that in particular, any affine invariant submanifold is a closed subset of $\mathcal{H}_1(\kappa)$ which is invariant under the $SL(2,\mathbb{R})$ action, and which in period coordinates looks like an affine subspace.

We also consider the entire stratum $\mathcal{H}(\kappa)$ to be an (improper) affine invariant submanifold.

The tangent space of an affine submanifold. Suppose $\mathcal{N}$ is an affine invariant submanifold. Then, by definition, in period coordinates the tangent bundle $T_{\mathcal{N}}$ of $\mathcal{N}$ is determined by a subspace $T_{\mathbb{C}}(\mathcal{N})$ of the vector space the ambient manifold is modeled on. Condition i) implies moreover that this subspace is of the form

$$T_{\mathbb{C}}(\mathcal{N}) = \mathbb{C} \otimes T_{\mathbb{R}}(\mathcal{N}),$$

where $T_{\mathbb{R}}(\mathcal{N}) \subset H^1(M, \Sigma, \mathbb{R})$. Let $p : H^1(M, \Sigma, \mathbb{R}) \to H^1(M, \mathbb{R})$ be the natural map. We can then consider the subspace $p(T_{\mathbb{R}}(\mathcal{N})) \subset H^1(M, \mathbb{R})$.

The Forni subspace. Let ν be a finite $SL(2,\mathbb{R})$-invariant measure on $\mathcal{H}_1(\kappa)$. For ν-almost all $x \in \mathcal{H}(\kappa)$, let $F(x) \subset H^1(M, \mathbb{R})$ be the maximal $SL(2,\mathbb{R})$-invariant subspace on which the Kontsevich-Zorich cocycle acts by isometries in the Hodge inner product (see §2 below). Then, the subspaces $F(x)$ form an $SL(2,\mathbb{R})$-invariant subbundle of the Hodge bundle.

For the Masur-Veech (i.e. Lebesgue) measure on $\mathcal{H}_1(\kappa)$, the Forni subspace $F(x) = \{0\}$ almost everywhere. However, there exist affine $SL(2,\mathbb{R})$-invariant measures ν for which $F(x) \neq 0$ for ν-almost-all x, see e.g. [Fo2], [FoM].

The main results. In this note, we prove the following:

Theorem 1.3. *Let ν be an affine measure on $\mathcal{H}_1(\kappa)$ and let $\mathcal{N}$ be affine submanifold on which ν is supported. Then,*

(a) *Except for a set of ν-measure 0, $F(x)$ is locally constant on $\mathcal{N}$.*

(b) *For ν-almost-all x, $p(T_{\mathbb{R}}(\mathcal{N}))(x)$ is orthogonal to $F(x)$ with respect to the Hodge inner product at x.*

(c) *For ν-almost-all x, $p(T_{\mathbb{R}}(\mathcal{N}))(x)$ is orthogonal to $F(x)$ with respect to the intersection form.*

As corollaries, we get the following:

Theorem 1.4. *Any affine manifold $\mathcal{N}$ is symplectic, in the sense that the intersection form is non-degenerate on $p(T_{\mathbb{R}}(\mathcal{N}))$.*

Theorem 1.5. *The Hodge bundle on any affine manifold $\mathcal{N}$ is semisimple, in the sense that any locally constant $SL(2,\mathbb{R})$-invariant subbundle has a complementary locally constant $SL(2,\mathbb{R})$-invariant subbundle.*

However the main motivation for this paper is that a somewhat more technical version of Theorem 1.3 where one does not assume that ν is an affine measure, see Theorem 7.2 below, is needed in [EMi] to complete the proof of the fact that all $SL(2,\mathbb{R})$-invariant measures are affine. (Theorem 7.2 is only needed in [EMi] in the presence of relative homology).

Acknowledgement. The authors would like to thank Alex Wright for making several useful suggestions, in particular regarding the formulation of Theorem 1.3 and Theorem 1.5.

2 The Kontsevich-Zorich Cocycle

Algebraic Hulls. The algebraic hull of a cocycle is defined in [Zi2]. We quickly recap the definition: Suppose a group G acts on a space X, preserving a measure ν, and suppose H is an an $\mathbb{R}$-algebraic group. Let $A : G \times X \to H$ be a cocycle, i.e. A is a measurable map, such that $A(g_1 g_2 x) = A(g_1, g_2 x)A(g_2, x)$. We say that the $\mathbb{R}$-algebraic subgroup H' of H is the *algebraic hull* of A if H' is the smallest $\mathbb{R}$-algebraic subgroup of H such that there exists a measurable map $C : X \to H$ such that

$$C(gx)^{-1}A(g,x)C(x) \in H' \qquad \text{for almost all } g \in G \text{ and almost all } x \in X.$$

It is shown in [Zi2] that the algebraic hull exists and is unique up to conjugation.

Period Coordinates. Let $\Sigma \subset M$ denote the set of zeroes of ω. Let $\{\gamma_1, \dots, \gamma_k\}$ denote a symplectic $\mathbb{Z}$-basis for the relative homology group $H_1(M, \Sigma, \mathbb{Z})$. We can define a map $\Phi : \mathcal{H}(\kappa) \to \mathbb{C}^k$ by

$$\Phi(M,\omega) = \left(\int_{\gamma_1} \omega, \dots, \int_{\gamma_k} w \right)$$

The map Φ (which depends on a choice of the basis $\{\gamma_1, \dots, \gamma_k\}$) is a local coordinate system on (M, ω). Alternatively, we may think of the cohomology class $[\omega] \in H^1(M, \Sigma, \mathbb{C})$ as a local coordinate on the stratum $\mathcal{H}(\kappa)$. We will call these coordinates *period coordinates.*

The $SL(2,\mathbb{R})$-action and the Kontsevich-Zorich cocycle. We write $\Phi(M,\omega)$ as a 2 by n matrix x. The action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ in these coordinates is linear. We choose some fundamental domain for the action of the mapping class group, and think of the dynamics on the fundamental domain. Then, the $SL(2,\mathbb{R})$ action becomes

$$x = \begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix} \to gx = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix} A(g,x),$$

where $A(g,x) \in \mathrm{Sp}(2g,\mathbb{Z}) \ltimes \mathbb{R}^k$ is the *Kontsevich-Zorich cocycle*. Thus, $A(g,x)$ is change of basis one needs to perform to return the point gx to the fundamental

domain. It can be interpreted as the monodromy of the Gauss-Manin connection (restricted to the orbit of $SL(2,\mathbb{R})$).

The following theorem is essentially due to Forni [Fo], and Forni-Matheus-Zorich [FMZ]. It is stated as [EMi, Theorem A.6]. For a self-contained proof (which essentially consists of references to [FMZ]) see Appendix A of [EMi].

Theorem 2.1. *Let ν be an $SL(2,\mathbb{R})$-invariant measure. Then,*

(a) *The ν-algebraic hull $\mathcal{G}$ of the Kontsevich-Zorich cocycle is semisimple.*

(b) *On any finite cover of $\mathcal{H}_1(\kappa)$, each ν-measurable irreducible subbundle of the Hodge bundle is either symplectic or isotropic.*

The Forni subspace.

Definition 2.2 (Forni Subspace)**.** Let

$$F(x) = \bigcap_{g \in SL(2,\mathbb{R})} g^{-1}(\operatorname{Ann} B^{\mathbb{R}}_{gx}), \tag{2.1}$$

where for $(M,\omega) \in \mathcal{H}_1(\kappa)$ the quadratic form $B^{\mathbb{R}}_{\omega}(\cdot,\cdot)$ is as defined in [FMZ] by

$$B^{\mathbb{R}}_{\omega}(\alpha,\beta) = \int_M \alpha\beta\frac{\overline{\omega}}{\omega}.$$

Remark. The form $B^{\mathbb{R}}_{\omega}(\cdot,\cdot)$ measures the derivative of the corresponding period matrix entry along a Teichmüller deformation in the direction of the quadratic differential ω^2 (Ahlfors-Rauch variational formula).

It is clear from the definition, that as long as its dimension remains constant, $F(x)$ varies real-analytically with x.

Theorem 2.3. *Suppose ν is an $SL(2,\mathbb{R})$-invariant measure. Then the subspaces $F(x)$ where x varies over the support of ν form the maximal ν-measurable $SL(2,\mathbb{R})$-invariant isometric subbundle of the Hodge bundle.*

Proof. Let $F(x)$ be as defined in (2.1). Then, F is an $SL(2,\mathbb{R})$-invariant subbundle of the Hodge bundle, and the restriction of $B^{\mathbb{R}}_x$ to $F(x)$ is identically 0. Consequently, by [FMZ, Lemma 1.9], F is isometric.

Now suppose N is any other ν-measurable isometric $SL(2,\mathbb{R})$-invariant subbundle of the Hodge bundle. Then by [FMZ, Theorem 2], $N(x) \subset \operatorname{Ann} B^{\mathbb{R}}_x$. Since M is $SL(2,\mathbb{R})$-invariant, we have $N \subset F$. Thus F is maximal. $\square$

Theorem 2.4. *On any finite cover of $\mathcal{H}_1(\kappa)$, the following statements hold.*

(a) *The Forni subspace is symplectic, and its symplectic complement $F^{\dagger}$ coincides with its Hodge complement $F^{\perp}$.*

(b) *Any invariant subbundle of $F^\perp$ is symplectic, and the restriction of the Kontsevich-Zorich cocycle to any invariant subbundle of $F^\perp$ has at least one non-zero Lyapunov exponent.*

Proof. See [EMi, Theorem A.9] (the proof of which consists of references to [FMZ]). □

Remark. In view of Theorem 2.4, the Forni subspace corresponds to the maximal compact factor of the algebraic hull of Kontsevich-Zorich cocycle over the action of $SL(2,\mathbb{R})$. By definition, all Lyapunov exponents on the Forni subspace are zero. However, the (non-$SL(2,\mathbb{R})$-invariant) zero-Lyapunov subspace of the Kontsevich-Zorich cocycle may be larger, see [FMZ] for an example.

3 The Real Analytic Envelope

In this section, we define a local real-analytic version of the Zariski closure of the support of an $SL(2,\mathbb{R})$-invariant measure. To study the Forni subspace, we must work in the real-analytic category, since the Forni subspace is a real-analytic object.

Let ν be an ergodic $SL(2,\mathbb{R})$-invariant measure on the stratum. We break up the stratum into charts, and consider each chart as a subset of $\mathbb{C}^n$.

Let $B(x,\epsilon)$ denote the ball centered at x of radius ϵ. Let $N(x,\epsilon)$ be the smallest real-analytic subset (in the sense of [N, Definition I.1]) of $B(x,\epsilon)$ such that $\nu(B(x,\epsilon)\cap N^c)=0$. Such an $N(x,\epsilon)$ exists by [N, Corollary V.2]. Note that $N(x,\epsilon)$ will be the empty set, if x is disjoint from the support of ν.

Let N be a real analytic set. A point $y\in N$ is called regular if, near y, N is a real-analytic submanifold of $\mathbb{C}^n$. Let N_{reg} denote the set of regular points of N and let N_{sing} denote the set of singular points.

Lemma 3.1. *There exists $\epsilon_0=\epsilon_0(x)$, such that the following conditions hold:*

i) For all $\epsilon<\epsilon_0$
$$N(x,\epsilon)=N(x,\epsilon_0)\cap B(x,\epsilon).$$

ii) If $S\subset B(x,\epsilon_0)$ is a real-analytic subset for which the inclusion $S_x\supset N_x(x,\epsilon_0)$ holds on the level of germs, then $S\supset N(x,\epsilon_0)$.

iii) The real-analytic set $N(x,\epsilon)$ has finitely many irreducible components.

iv) [N, Proposition III.5, p. 39] holds for $B(x,\epsilon_0)$, i.e. there is a non-zero analytic function δ_x on $B(x,\epsilon_0)$, such that the set of regular points $N_{reg}(x,\epsilon_0)$ contains $\{x:\delta_x(x)\neq 0\}$.

Proof. Claim i) follows from [N, Corollary V.1]. Claim ii) is a consequence of the Weierstrass preparation theorem, proven in [N, Theorem V.1, p. 98]. We have to shrink ϵ_0 somewhat more to achieve this.

The third claim is obvious on the level of germs ([N, Proposition III.1, p. 32]), i.e. $N_x(x, \epsilon_0) = \cup_{i=1}^k N_{x,i}$ with $N_{x,i}$ irreducible germs. On $B(x, \epsilon_0)$ by ii) there are irreducible analytic subsets N_i with germs $N_{x,i}$. Moreover, by ii) we have $N(x, \epsilon_0) = \cup_{i=1}^k N_i$. This proves iii).

The statement iv) is claimed in [N, Proposition III.5, p. 39] for irreducible germs. It can obviously be extended to a finite number of irreducible components taking the product of the corresponding functions δ_i on the components N_i. $\square$

Remark. The containment in iv) may be strict; see e.g. [N, Example 3, page 106].

Notation. Let $\epsilon_0 = \epsilon_0(x)$ be as in Lemma 3.1. We denote $B(x, \epsilon_0)$ by $U(x)$ and $N(x, \epsilon_0) \subset U(x)$ by $N(x)$.

The following lemma follows directly from the definition of $N(x)$ and Lemma 3.1.

Lemma 3.2. *Suppose $f : U(x) \to \mathbb{R}$ is a real analytic function such that $f(y) = 0$ for ν-almost-all $y \in N(x)$. Then f is identically zero on $N(x)$.*

Corollary 3.3. *Let ϵ_0 and $B(x, \epsilon_0)$ be as in Lemma 3.1. Then,*

$$\nu(N(x) \cap \{\delta_x \neq 0\}) > 0, \quad \textit{in particular} \quad \nu(N(x)_{reg}) > 0.$$

Proof. Suppose not. Then, ν is supported on a proper real analytic subset of $N(x)$, which contradicts Lemma 3.1 and the definition of $N(x)$. $\square$

Lemma 3.4. *The sets $N(x)$, $N(x)_{reg}$ and $N(x)_{sing}$ are $SL(2, \mathbb{R})$-equivariant in the following sense: suppose that $g \in SL(2, \mathbb{R})$, and let $U(g, x) = U(gx) \cap gU(x)$. Then $U(g, x)$ is an open neighborhood of gx, and on $U(g, x)$ we have*

$$N(gx) = gN(x), \quad N(gx)_{reg} = gN(x)_{reg}, \quad N(gx)_{sing} = gN(x)_{sing}.$$

Proof. Since ν is g-invariant, we have $N(gx) = gN(x)$ on $U(g, x)$ by the definition of $N(x)$. Since the $SL(2, \mathbb{R})$ action is smooth, the same argument shows that $N(gx)_{reg} = gN(x)_{reg}$. The final assertion follows from the fact that $N(x)_{sing} = N(x) \setminus N(x)_{reg}$. $\square$

In view of Lemma 3.4, $\dim N(x)$ is an $SL(2, \mathbb{R})$-invariant ν-measurable function on the stratum. Therefore, since ν is assumed to be ergodic, there exists a set Φ with $\nu(\Phi) = 1$ and $d \in \mathbb{Z}$ such that $\dim N(x) = d$ for all $x \in \Phi$.

Lemma 3.5. *Suppose $N \subset U$ is a real-analytic set defined on an open set $U \subset \mathbb{C}^n$, $y \in N_{reg}$, $V \subset U$ is a neighborhood of y satisfying Lemma 3.1, and N' is a real-analytic set such that $N' \cap V \subset N \cap V$ and $\dim N' = \dim N$. Then, $N' \cap V = N \cap V$.*

Proof. On the level of germs at y this is precisely [N, Proposition 7, p. 41]. By our choice of neighborhoods according to Lemma 3.1 ii), an equality of the germs implies equality of the analytic sets that induce the germs. $\square$

Proposition 3.6. *For ν-almost all y in the stratum, we have $y \in N(y)_{reg}$.*

Proof. Let

$$E = \{y \ : \ y \in N(y)_{reg}\}.$$

By Lemma 3.4 the set E is $SL(2,\mathbb{R})$-invariant. Therefore, by ergodicity, it is enough to show that $\nu(E) > 0$. We start with some y_0 such that $N(y_0)$ is not empty.

Choose an arbitrary $x \in N(y_0) \cap \{\delta_{y_0} \neq 0\} \cap \Phi$. By Corollary 3.3, for the neighborhood $U(x)$ of x we know that $\nu(N(x)_{reg}) > 0$. Therefore

$$\nu(N(x)_{reg} \cap \Phi) > 0. \tag{3.1}$$

Suppose $y \in N(x) \cap \{\delta_x \neq 0\} \cap \Phi$. Choose a neighborhood V of y with $V \subset U(y) \cap U(x)$. Then, by Lemma 3.1 i) we have $N(y) \cap V \subset N(x) \cap V$. Also since both $x \in \Phi$ and $y \in \Phi$, we have $\dim N(x) = \dim N(y)$. Therefore by Lemma 3.5, we have $N(y) \cap V = N(x) \cap V$. Hence we may take $\delta_x = \delta_y$ and conclude

$$N(y) \cap \{\delta_y \neq 0\} \cap V = N(x) \cap \{\delta_x \neq 0\} \cap V$$

as well as $N(y)_{reg} \cap V = N(x)_{reg} \cap V$. Since y was assumed to be in $N(x) \cap \{\delta_y \neq 0\}$, we have $y \in N(y) \cap \{\delta_y \neq 0\}$. Thus, $y \in E$. We have shown that

$$N(x) \cap \{\delta_x \neq 0\} \cap \Phi \subset E.$$

Therefore, by (3.1), $\nu(E) > 0$. □

Proposition 3.7. *For ν-almost all x, $N(x) \subset U(x)$ is affine. In particular, $TN(x)$ is preserved by the complex structure J.*

Outline of Proof. The relevant parts of the proof in [AG] apply. The "second step" and "third step" can be done for almost all $y \in N(x)$. Then by Lemma 3.2, the conclusion of the "third step" holds for all $y \in N(x)$. Then the "fourth step" proceeds exactly as in [AG], and this shows that $N(x)$ is affine. □

4 The Forni subspace revisited.

Everything in this section is a local statement about the intersection of the tangent space to the real analytic envelope of ν and the Forni subspace. Since everything is local (say around x) we may assume by Proposition 3.7 that the real analytic envelope $N = N(x)$ of ν is affine.

Since the form $B_x^{\mathbb{R}}(\cdot,\cdot)$ depends real-analytically on x, we may (for ν-almost all $x \in \mathcal{H}_1(\kappa)$) shrink $U(x)$ so that $\dim F(y)$ stays constant for all $y \in U(x)$. Then $F(y)$ depends real-analytically on $y \in U(x)$.

We have the following:

Lemma 4.1. *For ν-almost all x there exists a neighborhood $U(x)$ such that for all $y \in U(x)$ the following hold:*

(a) *The subspace $F^{\perp}(y)$ defined as the orthogonal complement of $F(y)$ using the Hodge inner product is $SL(2,\mathbb{R})$-invariant.*

(b) *For $v \in F(y)$, and $w \in F^{\perp}(y)$, $\langle v, w\rangle = 0$, where $\langle \cdot, \cdot \rangle$ denotes the symplectic form.*

(c) *If $y = a + bi$, then the space spanned by a and b is contained in $F^{\perp}(y)$.*

(d) *The restriction of the symplectic form to $F(y)$ is non-degenerate.*

Proof. Since $F(y)$ depends real-analytically on y and the Hodge inner product does, also $F^{\perp}(y)$ depends real-analytically on y. Therefore, by Lemma 3.2 it is enough to show (a)-(d) for y in the support of ν. The statements (a), (b), (d) follow immediately from Theorem 2.4. Also (c) is also clear on the support of ν since F has to be symplectically orthogonal to the $SL(2,\mathbb{R})$ orbit. $\square$

5 A connection on the unstable leaf.

Period Coordinates and the geodesic flow. Locally around a point $x = (M,\omega) \in \mathcal{H}_1(\kappa)$ we identify the tangent space to the stratum with

$$H^1(M,\Sigma,\mathbb{C}) = H^1(M,\Sigma,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

In this identification $SL(2,\mathbb{R})$ acts on $\mathbb{C} \cong \mathbb{R}^2$. Let $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ denote the geodesic flow. We identify leaves of the unstable foliation of g_t with a subspace of $H^1(M,\Sigma,\mathbb{R})$. More precisely, if $x = a + bi$, then the unstable leaf through x is identified with the span of b and $H_\perp$, where $H_\perp$ is the symplectic complement to the span of a and b. Similarly, we identify the leaves of the stable foliation with the span of ai and $iH_\perp$ in $H^1(M,\Sigma,\mathbf{i}\mathbb{R})$.

Let $A(x,t)$ denote the Kontsevitch-Zorich cocycle (i.e. the action of the Gauss-Manin connection on the real relative cohomology $H^1(M,\Sigma,\mathbb{R})$). We view $A(x,t)$ as an endomorphism of $H^1(M,\Sigma,\mathbb{R})$, whereas [AG] use the parallel transport $Dg_t : H^1(M,\Sigma,\mathbb{R}) \to H^1(g_tM,\Sigma,\mathbb{R})$. The two maps differ by a scaling factor e^t.

Lyapunov Exponents. Let $\{\lambda_i \ : \ i \in \Lambda\}$ denote the Lyapunov exponents. Let

$$\Lambda^- = \{i \in \Lambda \ : \ \lambda_i < 0\}, \qquad \Lambda^+ = \{i \in \Lambda \ : \ \lambda_i > 0\}.$$

For a generic point x in the support of ν, and let $E_i(x)$ be the Lyapunov subspace (at x) corresponding to the Lyapunov exponent λ_i.

Suppose $x \in \operatorname{supp}\nu$, and $s \in H^1(M,\Sigma,\mathbb{R})$. For s sufficiently small $x+s$ is a well-defined point in $\mathcal{H}_1(\kappa)$ and is in the same leaf of the unstable foliation as x. We do not assert that $x+s$ is in the support of ν. In our local coordinates,

$$g_t(x+s) = g_tx + e^tA(x,t)s.$$

A flat connection on the unstable leaf. Suppose $v(x) \in F(x)$. Since for small s, $F(x+s)$ and $F^{\perp}(x)$ are complementary, we may write

$$v(x) = v_x(x+s) - w(x,s), \qquad \text{where } v_x(x+s) \in F(x+s),\ w(x,s) \in F^{\perp}(x).$$

Then,

$$v_x(x+s) = v(x) + w(x,s). \tag{5.1}$$

Thus we have a linear map $P^+(x, x+s) : F(x) \to F(x+s)$ such that

$$P^+(x, x+s)v(x) = v_x(x+s).$$

In view of Lemma 5.1 below, it is easy to see that the map $P^+(x, x+s)$ coincides with the restriction to the Forni subspace of the measurable flat connection P^+ defined in [EMi, 4.2]. The main difference is that in our context, the map $P^+(x, x+s)$ depends real-analytically on x and $x+s$.

Fix $\beta > 0$ and let K'_β be the set where all saddle connections have length at least β. Then the Hodge inner product $Q_x(\cdot,\cdot)$ is uniformly continuous on $K'_{\beta/2}$. We can pick a compact subset $K_\beta \subset K'_\beta$ of positive measure such that the subspaces $F(x)$ are uniformly continuous on K_β. We denote by $||\cdot||$ the Hodge norm, given by $\|v\|_x = Q_x(v,v)^{1/2}$.

Lemma 5.1. *Suppose x is ν-generic. Then,*

(a) *The vector $P^+(x, x+s)v(x)$ depends real-analytically on s.*

(b) *$P^+(x, x+s)v(x)$ is the unique vector in $F(x+s)$ such that for any sequence $t_n \to -\infty$ with $g_{t_n}x \in K_\beta$, we have*

$$\|A(x,t_n)P^+(x,x+s)v(x) - A(x,t_n)v(x)\| \to 0.$$

(c) *For any s_1 and s_2 in the unstable leaf through x, we have*

$$P^+(x+s_1, x+s_1+s_2)P^+(x, x+s_1)v(x) = P^+(x, x+s_1+s_2)v(x).$$

Proof. The statement (a) is clear from the definition of $P^+(x, x+s)$ and the fact that $F(x+s)$ is analytic in s. To see (b), we will apply the geodesic flow to (5.1). Let $y_n = g_{t_n}x$. Let $\xi_n \in H^1(M, \Sigma, \mathbb{R})$ be such that

$$g_{t_n}(x+s) = y_n + \xi_n.$$

We have $\xi_n \to 0$ as $t_n \to -\infty$. We have

$$A(x,t_n)v(x) = A(x,t_n)P^+(x,x+s)v(x) + A(x,t_n)w(x,s).$$

We may start with $v(x)$ of norm $||v(x)|| = 1$ and thus by our convention on the Kontsevich-Zorich cocycle

$$\|A(x,t_n)v(x)\|_{y_n} = 1, \quad \|A(x,t_n)P^+(x,x+s)v(x)\|_{y_n+\xi_n} = 1. \tag{5.2}$$

Also, $A(x,t_n)v(x) \in F(y_n)$ and

$$A(x,t_n)P^+(x,x+s)v(x) \in F(y_n+\xi_n), \quad A(x,t_n)w(x,s) \in F^\perp(y_n).$$

Since $\xi_n \to 0$ and $y_n \in K_\beta$, we have $F(y_n+\xi_n) \to F(y_n)$. This, combined with (5.2) implies that

$$A(x,t_n)w(x,s) \to 0, \tag{5.3}$$

which proves (b). Now, (b) implies that $\lim_{t_n \to -\infty} A(x,t_n)v_x(x+s)$ is independent of s, which proves (c). $\square$

Notation. In view of (c), when there is no potential for confusion, we denote $v_x(x+s) = P^+(x,x+s)v(x)$ simply by $v(x+s)$.

Remark. The equation (5.3) implies that

$$v(x+s) = v(x) + v_0(x,s) + \sum_{i \in \Lambda^+} v_i(x,s) \tag{5.4}$$

where $v_0(x,s) \in E_0(x) \cap F^\perp(x)$, and for $i \in \Lambda^+$, $v_i(x,s) \in E_i(x) \subset F^\perp(x)$. The subspace $E_0(x) \cap F^\perp(x)$ may be non-empty since there can be zero Lyapunov exponents outside of the isometric subbundle F, see [FMZ] .

Lemma 5.2. (cf. [EMi, Proposition 4.4(b)]) *The parallel transport P^+ defined above preserves the Hodge inner product $Q(\cdot,\cdot)$ on F. In other words, for $v, w \in F(x)$,*

$$Q_{x+s}(v(x+s), w(x+s)) = Q_x(v(x), w(x))$$

Proof. Let t_n, y_n, ξ_n be as in the proof of Lemma 5.1. We have

$$Q_{y_n}(A(x,t_n)v(x), A(x,t_n)w(x)) = Q_x(v(x), w(x)) \tag{5.5}$$

and

$$Q_{y_n+\xi_n}(A(x,t_n)v(x+s), A(x,t_n)w(x+s)) = Q_{x+s}(v(x+s), w(x+s)) \tag{5.6}$$

We have $\xi_n \to 0$, and by Lemma 5.1 (b) we have

$$\|A(x,t_n)v(x+s) - A(x,t_n)v(x)\| \to 0, \qquad \|A(x,t_n)w(x+s) - A(x,t_n)w(x)\| \to 0.$$

Thus, the left-hand-sides of (5.5) and (5.6) approach each other as $t_n \to \infty$. Thus the right-hand-sides of (5.5) and (5.6) are equal. $\square$

6 A Formula for $P^+(x,x+s)$.

In this section, we derive an explicit formula, Lemma 6.4, for the parallel transport map P^+ defined in the previous section. The main tool is the real-analyticity of the connection P^+.

Let $A_i(x,t)$ denote the restriction of $A(x,t)$ to $E_i(x)$.

Lemma 6.1. *Let $v_0(x,s)$, $v_i(x,s)$ be as in (5.4). Suppose $t<0$ is such that $g_t x \in K_\beta$. Then for $i \in \Lambda^+ \cup \{0\}$ we have the Taylor expansion*

$$v_i(x,s) = A_i(x,t)^{-1} \sum_\alpha c_{i,\alpha} (e^t A(x,t)s)^\alpha, \tag{6.1}$$

where α is a multi-index, and and the $c_{i,\alpha} \in E_i(g_t x) \cap F^\perp(g_t x)$ are bounded independently of t.

Proof. Let $y = g_t x$, and let ξ be such that

$$g_t(x+s) = y + \xi, \qquad \text{where } \xi = e^t A(x,t)s.$$

Write

$$A(x,t)v(x+s) = w(y+\xi), \text{ so that } \quad \|w(y)\| = 1.$$

Since $w(y+\xi)$ depends real-analytically on ξ, we can Taylor expand

$$w_i(y+\xi) = w_i(y) + \sum_\alpha c_{i,\alpha}\xi^\alpha,$$

where $c_{i,\alpha} \in E_i(x)$, α is a multi-index and we use the standard multi-index notation. In particular $|\alpha|$ denotes the sum of the indices in α. Note that the coefficients $c_{i,\alpha}$ are uniformly bounded for $y \in K_\beta$ (in terms of i and α). Then, for $i \in \Lambda^+ \cup \{0\}$,

$$A_i(x,t)v_i(x,s) = \sum_\alpha c_{i,\alpha}(e^t A(x,t)s)^\alpha,$$

(and we used the fact that $v_i(0) = 0$). Now applying $A_i(x,t)^{-1}$ to both sides, we obtain (6.1). □

Remark. It is clear from Lemma 6.1 that $v_i(x,s)$ is a polynomial in s, i.e. for each $s \in H^1(M,\mathbb{R})$ the dependence of $v_i(x,s)$ for varying s in a fixed basis of E_i is polynomial in s. Indeed if $|\alpha|$ is sufficiently large (depending on the Lyapunov spectrum) then for any $t<0$ such that $g_t x \in K_\beta$, the coefficient of s^α in $v_i(x,s)$ is bounded by

$$\|A_i(x,t)^{-1}\|\|e^t A(x,t)\|^\alpha \le C\|A_i(x,t)^{-1}\|e^{(1-\lambda_2)|\alpha|t} \le Ce^{-t}e^{(1-\lambda_2)|\alpha|t}$$

where $\lambda_2 < 1$ is second Lyapunov exponent of the Kontsevich-Zorich cocycle. Thus, if $|\alpha| > 1/(1-\lambda_2)$, the right-hand-side tends to 0 as $t \to -\infty$.

Lemma 6.2. *Suppose $s \in \bigoplus_{j\in\Lambda^+} E_j(x)$. Then, $F(x+s) = F(x)$.*

Proof. Suppose $v(x) \in F(x)$, and for $i \in \Lambda^+ \cup \{0\}$ let $v_i(s)$ be as in (5.4). Write

$$s = \sum_{j\in\Lambda^+} s_j,$$

where $s_j \in E_j(x)$. Then, as $t \to -\infty$,

$$\|e^t A(x,t) s_j\| \approx e^{(1+\lambda_j)t} \|s_j\|$$

and

$$\|A(x,t)^{-1} c_{i,\alpha}\| \leq C e^{-\lambda_i t}, \quad \text{where } \lambda_i \leq 1.$$

It now follows from (6.1) that for $t < 0$ such that $g_t x \in K_\beta$,

$$\|v_i(x,s)\| \leq C \|A(x,t)^{-1}\| \max_{j \in \Lambda^+} \|e^t A(x,t) s_j\| \leq C \max_{j \in \Lambda^+} e^{(1-\lambda_i+\lambda_j)t} \|s_j\|$$

which tends to 0 as $t > -\infty$ since $1 - \lambda_i \geq 0$ and $\lambda_j > 0$. Thus, $v(x+s)$ is independent of the s_j for $j \in \Lambda^+$. $\square$

Let $p : H^1(M, \Sigma, \mathbb{R}) \to H^1(M, \mathbb{R})$ be the natural projection map. We recall that the Lyapunov spectrum of the Kontsevich-Zorich cocycle on $H^1(M, \Sigma, \mathbb{R})$ consists of the Lyapunov spectrum of the Kontsevich-Zorich cocycle on $H^1(M, \mathbb{R})$ union $\dim(\ker p)$ zeroes. It can also be easily shown that if λ is a nonzero Lyapunov exponent, $\tilde{E}_\lambda$ the Lyapunov subspace of $H^1(M, \Sigma, \mathbb{R})$ corresponding to λ, and E_λ the Lyapunov subspace corresponding to λ on $H^1(M, \mathbb{R})$, then $p(\tilde{E}_\lambda) = E_\lambda$, and p induces an isomorphism between these two subspaces.

Let $L(x) \subset H^1(M, \Sigma, \mathbb{R})$ be the smallest $SL(2, \mathbb{R})$-invariant subspace such that $p(L(x)) \supset F^\perp(x)$. We have

$$H^1(M, \Sigma, \mathbb{R}) = L(x) + F(x) + \ker p \tag{6.2}$$

This sum of subspaces need not be direct.

Lemma 6.3. *Suppose $x_0 \in \operatorname{supp} \nu$, and $x \in N(x_0)$. Then, for all $s \in L(x)$,*

$$F(x+s) = F(x).$$

Proof. For $x \in N(x_0)$, let $\mathcal{L}(x)$ denote the set of y in $N(x_0)$ such that $F(y) = F(x)$. Then $\mathcal{L}(x)$ is a real-analytic set which is $SL(2, \mathbb{R})$-equivariant. We can think of F as a function from N to the Grassmanian. We pick local coordinates on the Grassmanian, and write $F = (F_1, \ldots, F_k)$. Let r be such that DF has generically rank r on the support of ν. Then, by Lemma 3.2, $\operatorname{rank} DF \leq r$ on $N(x_0)$ (otherwise the support of ν would be contained in a proper real-analytic subset of $N(x_0)$), and by continuity, $\operatorname{rank} DF \geq r$ on an neighborhood of $x \in N$. Then, by the implicit function theorem, on a neighborhood of x in the support of ν, $\mathcal{L}(x)$ is regular. Then, the proof of Proposition 3.7 applies, and we have that $\mathcal{L}(x)$ is affine and defined over $\mathbb{R}$. Since being affine and being defined over $\mathbb{R}$ are real-analytically closed conditions, by Lemma 3.2, the same is true for all $x \in N(x_0)$. Thus there exists an $SL(2, \mathbb{R})$-invariant subspace $L(x) \subset H^1(M, \mathbb{R})$ such that $\mathcal{L}(x) = \{x + z \quad : \quad z \in \mathbb{C} \otimes L(x)\}$. By Lemma 6.2, $L(x)$ contains $\bigoplus_{i \in \Lambda^+} E_i(x)$. Since $L(x)$ is $SL(2, \mathbb{R})$-invariant, we must have $p(L(x)) \supset F^\perp(x)$. $\square$

Lemma 6.4. *Suppose $x \in N(x_0)$ is in the support of the measure ν. If $x = a + b\mathbf{i}$ (with $Area(x) = \langle a, b\rangle = 1$), we have for any $v(x) \in F(x)$ and any $x + s$ in the unstable leaf through x with $\langle p(s), p(b)\rangle = 0$,*

$$P^+(x, x + s)v(x) \equiv v(x + s) = v(x) + \langle v, p(s)\rangle p(b). \tag{6.3}$$

Proof. For s sufficiently small, we have in view of (6.2),

$$H^1(M, \Sigma, \mathbb{R}) = L(x + s) + F(x) + \ker p.$$

Therefore, we may write $s = s_1 + s_2$, where $s_1 \in L(x+s)$ and $s_2 \in F(x)+\ker p$. Then, by Lemma 6.3, $F(x + s) = F(x + s_2)$. Then, by the definition of v,

$$v(x + s) = v(x + s_2).$$

Then we can plug in into (6.1) with $s = s_2$. Note that $A(x, t)$ acts with zero Lyapunov exponents on $F + \ker p$. Let $v_i(x, s_2)$ be as in (5.4). Then,

$$\|v_i(x, s_2)\| \leq C \sum_\alpha \|A_i(x, t)^{-1}\| e^{t|\alpha|} s_2^\alpha \leq C \sum_\alpha e^{(-\lambda_i+|\alpha|)t} s_2^\alpha$$

Since $\lambda_i \leq 1$, we see that the coefficient of s_2^α for any $|\alpha| > 1$ tends to 0 as $t \to -\infty$ with $g_t x \in K_\beta$. Therefore all quadratic and higher order terms vanish. Also the only surviving linear term is with $\lambda_i = 1$. Note that the Lyapunov subspace on $H^1(M, \mathbb{R})$ corresponding to the Lyapunov exponent 1 is 1-dimensional, and $p(b)$ belongs to it. Hence,

$$v((x + s_1) + s_2) = v(x + s_1) + \Phi_x(s_2)p(b),$$

where Φ_x is a linear map. Thus,

$$v(x + s) = v(x) + \Phi_x(s_2)p(b).$$

But, we know that $v(x+s)$ has to be symplectically orthogonal to the $SL(2, R)$ orbit at $x + s$, i.e. the space spanned by $p(a + s)$ and $p(b)$. Thus,

$$0 = \langle p(a + s), v(x + s)\rangle = \langle p(a + s), v(x)\rangle + \Phi_x(s_2)\langle p(a + s), p(b)\rangle.$$

We have $\langle p(a), v(x)\rangle = 0$, $\langle p(s), p(b)\rangle = 0$, $\langle p(a), p(b)\rangle = 1$. Then,

$$\Phi_x(s_2) = -\langle p(s), v\rangle = -\langle p(s_2), v\rangle.$$

Thus (6.3) follows. ∎

7 The curvature of the connection.

We have defined a connection P^+ on unstable leaves. Similarly, there is a connection P^- on stable leaves. Since both of these connections are real-analytic, they can together define a connection P on the real-analytic envelope of the support of the measure. It turns out that this connection is no longer flat. The key calculation of this paper is the following proposition, which can be viewed as computing the curvature of the connection.

Proposition 7.1. *Suppose x_0 is in the support of ν , and let $N = N(x_0)$. Let TN be the tangent space to the affine manifold N. We have by Proposition 3.7, that $TN = \mathbb{C} \otimes T_{\mathbb{R}}N$ for some subspace $T_{\mathbb{R}}N \subset H^1(M, \Sigma, \mathbb{R})$. Then, for all $x \in N$,*

$$p(T_{\mathbb{R}}N) \subset F^{\perp}(x) \tag{7.1}$$

and $F(x)$ is locally constant on N.

Proof. Write $x = a + b\mathbf{i}$. Suppose x is in the support of ν. Suppose (7.1) fails. Choose y in the support of ν such that $y - x \in TN$ and $p(y - x) \notin \mathbb{C} \otimes F^{\perp}(x)$. Let $\delta \in T_{\mathbb{R}}N$ denote the real part of $y - x$. Then $\delta \notin F^{\perp}(x)$. Without loss of generality, we may assume that $p(\delta)$ is symplectically orthogonal to the $SL(2, \mathbb{R})$ orbit, i.e. $\langle p(\delta), p(a)\rangle = \langle p(\delta), p(b)\rangle = 0$.

Let $\epsilon > 0$ be arbitrary. Now let v be an arbitrary element of $F(x)$. We will now move v around a square

$$a + b\mathbf{i} \to (a + \delta) + b\mathbf{i} \to (a + \delta) + (b + \epsilon a)\mathbf{i} \to a + (b + \epsilon a)\mathbf{i} \to a + b\mathbf{i}.$$

In other words we compute

$$P^{-}(a + (b + \epsilon a)\mathbf{i}, a + b\mathbf{i})\, P^{+}((a + \delta) + (b + \epsilon a)\mathbf{i}, a + (b + \epsilon a)\mathbf{i})$$
$$P^{-}((a + \delta) + b\mathbf{i}, (a + \delta) + (b + \epsilon a)\mathbf{i})\, P^{+}(a + b\mathbf{i}, (a + \delta) + b\mathbf{i})v$$

- Step 1: moving from $a + b\mathbf{i}$ to $(a + \delta) + b\mathbf{i}$. Using (6.3) with $x = a + b\mathbf{i}$ and $s = \delta$ we get
$$v \to v + \langle v, p(\delta)\rangle p(b).$$

- Step 2: moving from $(a+\delta)+b\mathbf{i}$ to $(a+\delta)+(b+\epsilon a)\mathbf{i}$. Using (6.3) with the contracting and expanding directions reversed (and taking into account the sign change coming from the fact that (6.3) was derived assuming $\langle a, b\rangle = +1$) and $x = (a + \delta) + b\mathbf{i}$ and $s = \epsilon a$ we get
$$\begin{aligned} v + \langle v, p(\delta)\rangle p(b) &\to v + \langle v, p(\delta)\rangle p(b) - \langle v + \langle v, p(\delta)\rangle p(b), \epsilon p(a)\rangle p(a + \delta) \\ &= v + \langle v, p(\delta)\rangle p(b) + \epsilon\langle v, p(\delta)\rangle p(a + \delta). \end{aligned}$$

- Step 3: moving from $(a + \delta) + (b + \epsilon a)\mathbf{i}$ to $a + (b + \epsilon a)\mathbf{i}$. Using (6.3) with $x = (a + \delta) + (b + \epsilon a)\mathbf{i}$ and $s = -\delta$ we get
$$\begin{aligned} &v + \langle v, p(\delta)\rangle p(b) + \epsilon\langle v, p(\delta)\rangle p(a + \delta) \to v + \langle v, p(\delta)\rangle p(b) + \epsilon\langle v, p(\delta)\rangle p(a + \delta) \\ &\quad - \langle v + \langle v, p(\delta)\rangle p(b) + \epsilon\langle v, p(\delta)\rangle p(a + \delta), p(\delta)\rangle p(b + \epsilon a) \\ &\quad = v + \langle v, p(\delta)\rangle p(b) + \epsilon\langle v, p(\delta)\rangle p(a + \delta) - \langle v, p(\delta)\rangle p(b + \epsilon a) \\ &\quad = v + \epsilon\langle v, p(\delta)\rangle p(\delta). \end{aligned}$$

- Step 4: moving from $a + (b + \epsilon a)\mathbf{i}$ to $a + b\mathbf{i}$. Using (6.3) with $x = a + (b + \epsilon a)\mathbf{i}$ and $s = -\epsilon a$ (and taking into account the sign change coming from reversing a and b) we get
$$\begin{aligned} v + \epsilon\langle v, p(\delta)\rangle p(\delta) &\to v + \epsilon\langle v, p(\delta)\rangle p(\delta) + \langle v + \epsilon\langle v, p(\delta)\rangle p(\delta), \epsilon p(a)\rangle p(a) \\ &= v + \epsilon\langle v, p(\delta)\rangle p(\delta). \end{aligned}$$

Thus, moving around the square, we map v to $v+\epsilon\langle v, p(\delta)\rangle p(\delta)$. This contradicts Lemma 5.2. Therefore (7.1) holds. (In the case when $p(\delta) \notin F(x)$, then this computation shows that moving around the square does not preserve $F(x)$; this this is a contradiction to the definition of the connection P^+.) $\square$

We have proved:

Theorem 7.2. *There exists a subset Ψ of the stratum $\mathcal{H}_1(\kappa)$ with $\nu(\Psi) = 1$ such that for all $x \in \Psi$ there exists a neighborhood $U(x)$ such that for all $y \in U(x) \cap \Psi$ we have $p(y - x) \in F^{\perp}(x)$.*

Theorem 7.2 is used in [EMi] to complete the proof that any $SL(2,\mathbb{R})$ invariant measure is affine.

Proof of Theorem 1.3. Let $\mathcal{N}$ be an affine submanifold, and let ν be the affine measure supported on $\mathcal{N}$. Then for ν-almost all $x \in \mathcal{N}$, $N(x) = \mathcal{N}$, where $N(x)$ is as in §3. Then parts (a) and (b) of Theorem 1.3 follow immediately from Proposition 7.1. Also part (c) of Theorem 1.3 follows immediately from part (a) of Theorem 2.4. $\square$

Proof of Theorem 1.4. Let $\mathcal{N}$ be an affine submanifold, and let ν be the affine measure supported on $\mathcal{N}$. By Theorem 2.1 for ν-a.e. x, we have the following decomposition,

$$H^1(M,\mathbb{R}) = \bigoplus_{i=1}^{m} L_i(x) \oplus \bigoplus_{j=1}^{m'} V_j(x)$$

where all factors are irreducible and $SL(2,\mathbb{R})$-invariant, the $L_i(x)$ are symplectic and the $V_i(x)$ are isotropic. Note that this decomposition is not unique, if some of the summands are isomorphic as $SL(2,\mathbb{R})$-invariant bundles. Therefore, since $p(T_{\mathbb{R}}(\mathcal{N}))$ is an $SL(2,\mathbb{R})$-invariant bundle, we may (by Theorem 2.1) arrange the decomposition so that

$$p(T_{\mathbb{R}}(\mathcal{N}))(x) = \bigoplus_{i\in I} L_i(x) \oplus \bigoplus_{j\in J} V_j(x),$$

for some $I \subset \{1, \dots, m\}$ and $J \subset \{1, \dots, m'\}$. By Lemma 4.1, $V_j(x) \subset F(x)$. Therefore, by Proposition 7.1, $J = \emptyset$. Thus, $\mathcal{N}$ is symplectic. $\square$

Proof of Theorem 1.5. The proof is a lightly modified version of the proof of [EMi, Theorem A.6]. By Proposition 7.1 and Theorem 2.4 we have the locally constant decomposition

$$H^1(M,\mathbb{R}) = F^{\perp} \oplus F,$$

where both factors are $SL(2,\mathbb{R})$-invariant. Suppose L is a locally constant $SL(2,\mathbb{R})$-invariant subbundle of the Hodge bundle. Let $L^{\dagger}$ be the symplectic complement to L, and let $L_1 = L \cap L^{\dagger}$. Then, L_1 is locally constant. Also L_1 is isotropic, and therefore by [EMi, Theorem A.4], [EMi, Theorem A.5] and Theorem 2.3, $L_1 \subset F$.

Note that by Lemma 5.2 and Proposition 7.1, F is locally constant and the monodromy representation restricted to F has compact image. Therefore there exists a an $SL(2,\mathbb{R})$-invariant locally constant subbundle L_2 of F such that $F = L_1 \oplus L_2$. Let $L_3 = L_2 \oplus F^{\perp}$; then L_3 is also locally constant and $SL(2,\mathbb{R})$-invariant. Then,

$$L = L_1 \oplus (L \cap L_3), \qquad L^{\dagger} = L_1 \oplus (L^{\dagger} \cap L_3),$$

and

$$H^1(M,\mathbb{R}) = L_1 \oplus (L \cap L_3) \oplus (L^{\dagger} \cap L_3).$$

Thus $L^{\dagger} \cap L_3$ is an $SL(2,\mathbb{R})$-invariant locally constant complement to L. $\square$

References

[AG] Avila, A., Gouezel, S., *Small eigenvalues of the Laplacian in the moduli space of Abelian differentials, and mixing properties of the Teichmüller flow*. Preprint, `arXiv:1011.5472` (2010).

[EMi] A. Eskin, M. Mirzakhani. *Invariant and stationary measures for the $SL(2,\mathbb{R})$ action on Moduli space*. Preliminary version of preprint available at `http://www.math.uchicago.edu/~eskin`.

[Fo] G. Forni, *Deviation of ergodic averages for area-preserving flows on surfaces of higher genus*. Annals of Math., **155**, no. 1 (2002), 1–103.

[Fo2] G. Forni. On the Lyapunov exponents of the Kontsevich-Zorich cocycle. *Handbook of dynamical systems. Vol. 1B*, pp 549–580. Elsevier B. V., Amsterdam, (2006).

[FoM] G. Forni, C. Matheus, *An example of a Teichmüller disk in genus 4 with degenerate Kontsevich-Zorich spectrum*. Preprint, `arXiv:0810.0023` (2008).

[FMZ] G. Forni, C. Matheus, A. Zorich. *Lyapunov Spectrum of Invariant Subbundles of the Hodge Bundle*. Preprint, `arXiv:1112.0370` (2011).

[N] R. Narasimhan, *Introduction to the theory of analytic spaces*. Lecture Notes in Math. 25 Springer-Verlag, Berlin-New York (1966).

[Zi2] R. J. Zimmer. *Ergodic Theory and Semisimple Groups.* Birkhäuser: Boston, (1984).

[Zo] A. Zorich, *Flat Surfaces*. Frontiers in Number Theory, Physics, and Geometry. I. Berlin: Springer, (2006), 437–583.